SWSC Functions Oral Reference – Contest 2
Sample Oral Questions

1. Comment on the truth of each statement below:
   I. $y = -x^2$ is a function in terms of x.
   II. $y = |x| - 1$ is a function in terms of x.
   III. $y^2 = x$ is a function in terms of x.
   IV. $|y| = x - 1$ is a function in terms of x.
   Explain how you arrived to each answer.

2. f is a linear function with f(1) = 2. If $f(-x) = -f(x)$, find the value of k if $f(k) = 12$. Explain how you arrived to your answer.

3. If $f(x) = x + a(x+1) + b(x+2)$ for a and b real constants, $f(1) = 7$ and $f(0) = 2$. Find the value of $f(-1)$.

4. $g(x) = f(x) + 1$ and f is a linear function with y-intercept 0 and positive slope. If the area bounded by f, the x-axis, and the line x = 2 is k, find the area bounded by the graph of g, the axes, and the line x = 2 in terms of k. Explain how you arrived to your answer.

5. If $f(x) = x^2 - 2x + 1$ and $h(x) = 2x - 8$, explain how you would find the value of $h(f(3))$.

6. Let $f(x) = mx + b$ be a linear function. A lattice point is defined as an ordered pair in the xy-plane with integer values. Comment on the truth of each of the following statements.
   I. f will pass through infinitely many lattice points if m is rational, not equal to zero.
   II. f will pass through infinitely many lattice points if m is irrational.
   III. f will pass through infinitely many lattice points if m = 0.
   Explain how you arrived to each answer.

7. If $f(1) = 4$ and $f(x) = 2f(x-1)$, explain how to find the value of $f(5)$.

8. If $f(x+5) = 4x^2 + 43x + 122$ and $f(x) = Ax^2 + Bx + C$, find the value of A + B + C. Explain how you arrived to your answer.

9. How many negative integral values does the range of $f(x) = \dfrac{16 - x^2}{3}$ have for a restricted domain $[0, 7]$? Explain how you arrived to your answer.

10. Comment on the truth of each of the following statements:

I. The discriminant of the function $f(x) = 3x^2 + 9x + 10$ is $\sqrt{39}$.

II. The function $q(x) = 2x^2 + 10x - 13$ has two complex conjugate roots.

III. The function $g(x) = x^2 - 13$ has two distinct roots.

11. Comment on the truth of each of the following statements:

I. The axis of symmetry of the function $q(x) = \frac{1}{2}x^2 - 2x + 5$ is the line x = 2.

II. The vertex of the function $f(x) = \frac{3}{4}x^2 - 2x + 3$ is $\left(\frac{4}{3}, 3\right)$.

III. The maximum value of the function $g(x) = \frac{-1}{3}x^2 + 6x$ is 27.

12. For how many integer values of x is $f(x) < 0$ if $f(x) = x^2 - 3x - 4$? Explain how you arrived to your answer.

13. Given the relations $f = \{(0, -3), (2, 5), (-1, 1), (4, 2)\}$, $g = \{(-1, 2), (1, 4), (4, 3), (0, -1)\}$, and $h = \{(4, 2), (1, 0), (-3, 4), (3, -1)\}$, find the relation of ordered pairs that represents $f(g(h(x)))$.

14. Allen and Beatrice are attempting to find the zeros of the quadratic function f(x). Allen writes down the equation but makes a mistake when copying down the constant term and gets -3 and 6 for the zeros. Beatrice also makes an error when copying the x-term and obtains -1 and 10 for the zeros. Assuming that both students did their work correctly (and just miscopied), what are the zeros of f(x)? Explain how you arrived to your answer.

15. If $h(x) = |x - 3|$, explain why $h(|x|) = -x + 3$ for all values of x less than –5.

16. If $f(x) = \begin{cases} |2x + 1| & |x| < 3 \\ 4 - x^2 & |x| \geq 3 \end{cases}$, show that the value of $f(f(-2))$ is –5 .

17. Let $a(x) = |x - 1|$ and $b(x) = |3x + 4|$. Let $c(x) = a(x) - b(x)$ when $0 < x < 1$. If $c(x) = mx + n$, find the value of m + n. Explain how you arrived to your answer.

18. The graphs of the functions $f(x) = x^2 - 4x + 3$ and $g(x) = x - 1$ intersect at what points? Explain how you arrived to your answer.

19. Given $f(x) = \frac{x - 2}{3}$, $g(x) = 14 - 7x$, $h(x) = \pi$, and $k(x) = x^2$, which of these functions are strictly increasing functions of x? Explain how you arrived to your answer.

20. The vertex of $f(x) = x^2 - 8x + C$ will be a point on the x-axis for what value of C? Explain how you arrived to your answer.

21. If the function f(x) is shifted to the right three coordinates and down six coordinates, the resulting function can be written in the form $f(x+a)+b$. Find the value of a + b. Explain how you arrived to your answer.

22. The function f, where $f(x)=(1+x)^2$, is defined for $-2 \le x \le 2$. Using interval notation, state the range of f. Explain how you arrived to your answer.

23. Let $f(x)=\begin{cases} x-2 & x>1 \\ 4-x & x\le 1 \end{cases}$. If $a>1$ and $b<1$, $f(a)=f(b)=3$, find the value of $|a-b|$. Explain how you arrived to your answer.

24. The graph of $f(x)=3x^2-12x+11$ is translated 3 units to the right and 5 units up to make a new function $g(x)$. If $g(x)=3(x-B)^2+D$, find the values of B and D. Explain how you arrived to your answer.

25. Find the minimum value of $f(x)=(x+3)(x-1)$. Explain how you arrived to your answer.

SWSC Functions Oral Reference – Contest 2
Sample Oral Questions w/Solutions

1. Comment on the truth of each statement below:
   I. $y = -x^2$ is a function in terms of x.
   II. $y = |x| - 1$ is a function in terms of x.
   III. $y^2 = x$ is a function in terms of x.
   IV. $|y| = x - 1$ is a function in terms of x.
   Explain how you arrived to each answer.

   Solution: I is a parabola opening downward…is a function. II is an absolute value function. III is a parabola opening right, but is not a function since when x = 4, y could be 2 or –2. IV is not a function since when x = 3, y could be 2 or –2.

2. f is a linear function with f(1) = 2. If $f(-x) = -f(x)$, find the value of k if $f(k) = 12$. Explain how you arrived to your answer.

   Solution: Since $f(-x) = -f(x)$, f must pass through the origin. Thus, f(x) = mx. Using the points f(1) = 2 and f(0) = 0, m = 2. Therefore, f(x) = 2x. 2x = 12 when x = 6, k = 6.

3. If $f(x) = x + a(x+1) + b(x+2)$ for a and b real constants, $f(1) = 7$ and $f(0) = 2$. Find the value of $f(-1)$.

   Solution: $\begin{cases} f(1): 1 + 2a + 3b = 7 \\ f(0): a + 2b = 2 \end{cases} \rightarrow \begin{cases} 2a + 3b = 6 \\ a + 2b = 2 \end{cases}$. Solving the system yields a = 6 and b = -2. Therefore, $f(-1) = -1 + 6(-1+1) - 2(-1+2) = -3$

4. $g(x) = f(x) + 1$ and f is a linear function with y-intercept 0 and positive slope. If the area bounded by f, the x-axis, and the line x = 2 is k, find the area bounded by the graph of g, the axes, and the line x = 2 in terms of k. Explain how you arrived to your answer.

   Solution: g is raised 1 unit from f, so we have the area of f added to a rectangle of dimensions 2 by 1 right above the x-axis. Therefore the area bounded by the graph of g is k + 2.

5. If $f(x) = x^2 - 2x + 1$ and $h(x) = 2x - 8$, explain how you would find the value of $h(f(3))$.

   Solution: Since $f(3) = 3^2 - 2(3) + 1 = 4$, we must evaluate $h(4)$. $h(4) = 2(4) - 8 = 0$.

6. Let $f(x) = mx + b$ be a linear function. A lattice point is defined as an ordered pair in the xy-plane with integer values. Comment on the truth of each of the following statements.
I. f will pass through infinitely many lattice points if m is rational, not equal to zero.
II. f will pass through infinitely many lattice points if m is irrational.
III. f will pass through infinitely many lattice points if m = 0.
Explain how you arrived to each answer.

Solution:
I. False – This is true only if the y-intercept is rational.
II. False – If a line with irrational slope were to pass through even two lattice points, say (a, b) and (c, d), then the slope of the line would be $\frac{d-b}{c-a}$, which must be rational, creating a contradiction.
III. False – This is true only if the y-intercept is rational.

7. If $f(1) = 4$ and $f(x) = 2f(x-1)$, explain how to find the value of $f(5)$.

Solution: $f(2) = 2f(1) \rightarrow f(3) = 2^2 f(1) \rightarrow f(4) = 2^3 f(1) \rightarrow f(5) = 2^4 f(1) = 16(4) = 64$

8. If $f(x+5) = 4x^2 + 43x + 122$ and $f(x) = Ax^2 + Bx + C,$ find the value of A + B + C. Explain how you arrived to your answer.

Solution: $f(x+5)$ also equals $A(x+5)^2 + B(x+5) + C$, or $f(x+5) = 25A + 5B + C + 10Ax + Bx + Ax^2$. Rearranging results in $Ax^2 + (10A + B)x + (25A + 5B + C) = 4x^2 + 43x + 122$. Since $Ax^2 = 4x^2$, A = 4. Therefore $10(4) + B = 43 \rightarrow B = 3$ and $25(4) + 5(3) + C = 122 \rightarrow C = 7$. A + B + C = 4 + 3 + 7 = 14.

9. How many negative integral values does the range of $f(x) = \frac{16 - x^2}{3}$ have for a restricted domain $[0, 7]$? Explain how you arrived to your answer.

Solution: $f(x) = \frac{(4-x)(4+x)}{3}$ is negative on the intervals $(-\infty, -4) \cup (4, \infty)$. Since $f(7) = -11$, there are 11 integral values for the restricted domain [0, 7].

10. Comment on the truth of each of the following statements:
I. The discriminant of the function $f(x) = 3x^2 + 9x + 10$ is $\sqrt{39}$.
II. The function $q(x) = 2x^2 + 10x - 13$ has two complex conjugate roots.
III. The function $g(x) = x^2 - 13$ has two distinct roots.

Solution:
I. The discriminant is $9^2 - 4(3)(10) = -39$, not $\sqrt{39}$.
II. Since the discriminant is $10^2 - 4(2)(-13) > 0$ the two roots are real, not imaginary/complex.
III. The roots are $\sqrt{13}$ and $-\sqrt{13}$, so this statement is true.

11. Comment on the truth of each of the following statements:

I. The axis of symmetry of the function $q(x) = \frac{1}{2}x^2 - 2x + 5$ is the line x = 2.

II. The vertex of the function $f(x) = \frac{3}{4}x^2 - 2x + 3$ is $\left(\frac{4}{3}, 3\right)$.

III. The maximum value of the function $g(x) = \frac{-1}{3}x^2 + 6x$ is 27.

Solution:

I. $q(x) = \frac{1}{2}(x^2 - 4x + 4) + 5 - 2 = \frac{1}{2}(x-2)^2 + 3$. Yes, the axis of symmetry is x – 2 = 0 or x = 2.

II. $f(x) = \frac{3}{4}x^2 - 2x + 3 = \frac{3}{4}\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + 3 - \frac{4}{3} = \frac{3}{4}\left(x - \frac{4}{3}\right)^2 + \frac{5}{3}$. No, the vertex is $\left(\frac{4}{3}, \frac{5}{3}\right)$.

III. $g(x) = \frac{-1}{3}x^2 + 6x = \frac{-1}{3}(x^2 - 18x + 81) + 27 = \frac{-1}{3}(x-9)^2 + 27$. Yes, the maximum value is 27.

12. For how many integer values of x is $f(x) < 0$ if $f(x) = x^2 - 3x - 4$? Explain how you arrived to your answer.

Solution: $x^2 - 3x - 4 < 0 \rightarrow (x-4)(x+1) < 0$ is true on the interval $-1 < x < 4$. The integers 0, 1, 2, 3 are included on this interval. Therefore, there are four values of x for which $f(x) < 0$.

13. Given the relations $f = \{(0,-3),(2,5),(-1,1),(4,2)\}$, $g = \{(-1,2),(1,4),(4,3),(0,-1)\}$, and $h = \{(4,2),(1,0),(-3,4),(3,-1)\}$, find the relation of ordered pairs that represents $f(g(h(x)))$.

Solution: $g(h(x)) = \{(3,2),(1,-1),(-3,3)\}$. Therefore, $f(g(h(x))) = \{(3,5),(1,1)\}$.

14. Allen and Beatrice are attempting to find the zeros of the quadratic function f(x). Allen writes down the equation but makes a mistake when copying down the constant term and gets -3 and 6 for the zeros. Beatrice also makes an error when copying the x-term and obtains -1 and 10 for the zeros. Assuming that both students did their work correctly (and just miscopied), what are the zeros of f(x)? Explain how you arrived to your answer.

Solution: Allen's function was $f(x) = (x+3)(x-6) = x^2 - 3x - 18$. While the -18 is incorrect, the x^2 and $-3x$ were correctly copied. Beatrice's function was $f(x) = (x+1)(x-10) = x^2 - 9x - 10$. While the -9 is incorrect, the x^2 and -10 were correctly copied. Therefore, the original function was $f(x) = x^2 - 3x - 10$. The zeros of f(x) are -2 and 5.

15. If $h(x) = |x-3|$, explain why $h(|x|) = -x+3$ for all values of x less than –5.

Solution: Since $|x| = -x$ for all x < 0, $h(x) = |-x+3|$ when x < –5. Since $-x-3$ is always positive when x < –5, $h(|x|) = -x+3$.

16. If $f(x) = \begin{cases} |2x+1| & |x| < 3 \\ 4-x^2 & |x| \ge 3 \end{cases}$, show that the value of $f(f(-2))$ is –5 .

Solution: Since $|-2| < 3$, $f(-2) = |2(-2)+1| = 3$. Since $|3| \ge 3$, $f(3) = 4-3^2 = -5$.

17. Let $a(x) = |x-1|$ and $b(x) = |3x+4|$. Let $c(x) = a(x) - b(x)$ when $0 < x < 1$. If $c(x) = mx+n$, find the value of m + n. Explain how you arrived to your answer.

Solution: If $0 < x < 1$, $a(x) = |x-1| = 1-x$ and $b(x) = |3x+4| = 3x+4$. Therefore, $c(x) = (1-x)-(3x+4) = -4x-3$. $m+n = -4+(-3) = -7$

18. The graphs of the functions $f(x) = x^2-4x+3$ and $g(x) = x-1$ intersect at what points? Explain how you arrived to your answer.

Solution: The points of intersection will occur when $x^2-4x+3 = x-1 \rightarrow x^2-5x+4 = 0 \rightarrow$ $(x-1)(x-4) = 0$. When x = 4, y = 3. When x = 1, y = 0. Therefore, the functions intersect at (4, 3) and (1, 0).

19. Given $f(x) = \frac{x-2}{3}$, $g(x) = 14-7x$, $h(x) = \pi$, and $k(x) = x^2$, which of these functions are strictly increasing functions of x? Explain how you arrived to your answer.

Solution: Since f is linear with a positive slope, it is strictly increasing. Since g is linear with a negative slope, it is not strictly increasing. Since h is a constant function, it is not strictly increasing. Since k is a quadratic function, it is not strictly increasing. Therefore f is the only strictly increasing function.

20. The vertex of $f(x) = x^2-8x+C$ will be a point on the x-axis for what value of C? Explain how you arrived to your answer.

Solution: $f(x) = (x^2-8x+16)+C-16 = (x-4)^2+C-16$. The vertex will be a point on the x-axis if the vertical coordinate is zero, or C – 16 = 0. Therefore C = 16.

21. If the function f(x) is shifted to the right three coordinates and down six coordinates, the resulting function can be written in the form $f(x+a)+b$. Find the value of a + b. Explain how you arrived to your answer.

Solution: A function shifted to the right three coordinates is $f(x-3)$. Shifting the function down six coordinates results in $f(x-3)-6$. The value of a + b is $-3+(-6)=-9$.

22. The function f, where $f(x)=(1+x)^2$, is defined for $-2 \le x \le 2$. Using interval notation, state the range of f. Explain how you arrived to your answer.

Solution: f(x) is a quadratic function with vertex (-1, 0), and opening upward. Therefore, the minimum value of the range is 0. Since $f(-2)=1$ and $f(2)=9$, the range is $[0, 9]$.

23. Let $f(x)=\begin{cases} x-2 & x>1 \\ 4-x & x \le 1 \end{cases}$. If $a>1$ and $b<1$, $f(a)=f(b)=3$, find the value of $|a-b|$. Explain how you arrived to your answer.

Solution: $x-2=3$ at x = 5, and $4-x=3$ at x = 1. $|5-1|=4$

24. The graph of $f(x)=3x^2-12x+11$ is translated 3 units to the right and 5 units up to make a new function $g(x)$. If $g(x)=3(x-B)^2+D$, find the values of B and D. Explain how you arrived to your answer.

Solution: $f(x)=3(x^2-4x+4)+11-12=3(x-2)^2-1$. Translating 3 units to the right and 5 units up results in $g(x)=3(x-2-3)^2-1+5=3(x-5)^2+4$. Therefore B = 5 and D = 4.

25. Find the minimum value of $f(x)=(x+3)(x-1)$. Explain how you arrived to your answer.

Solution: $f(x)=x^2+2x-3=(x^2+2x+1)-3-1=(x+1)^2-4$. Therefore, the minimum value of f(x) is –4.

SWSC Functions Oral Reference – Contest 4
Sample Oral Questions

1. Find the polynomial equation of the cubic function f(x) which has zeros 4 and $3+\sqrt{2}$. Explain how you arrived to your answer.

2. Two of the roots of the function $g(x)=x^3+3x^2+kx-12$ are real and unequal but have the same absolute value. Find the value of k. Explain how you arrived to your answer.

3. If $f(x)=x^3-10x^2+Ax+B$, f has one root at x = 2, and A + B = 9, find the value of f(1). Explain how you arrived to your answer.

4. For what values of x is f(x) > 0 if $f(x)=x^3-7x+6$? Express your answer using interval notation. Explain how you arrived to your answer.

5. Through what quadrant(s) does the slant asymptote of $g(x)=\dfrac{x^2+6x+1}{x+2}$ pass through if it is graphed in the Cartesian coordinate system? Explain how you arrived to your answer.

6. If the graph of $f(x)=\dfrac{x^3+6x^2+3x-10}{x^2-6x+5}$ contains 'a' vertical asymptotes, 'b' horizontal asymptotes, and 'c' slant asymptotes, find the values of a, b, and c. Explain how you arrived to your answer.

7. $f(x+1)=\dfrac{x^2+9x+12}{2x+7}$. If $f(3-x)$ is written in the form $f(3-x)=\dfrac{Ax^2+Bx+C}{Ax+B}$, find $f(3-x)$. Explain how you arrived to your answer.

8. For what values of x is the function $h(x)<0$ if $h(x)=\dfrac{3x}{x(x^2-16)}$? Express your answer using interval notation, and explain how you arrived to your answer.

9. If $g(x)=\dfrac{x^3-3x^2-9x+27}{x^2+2x-15}$:
   a. Find the coordinates of the hole in the graph.
   b. Find the number of zeros of g.
   c. Find the x-intercept of the slant asymptote
   Explain how you arrived to each answer.

10. Rewrite the function $f(x)=\left||x-3|-4\right|$ as a piecewise function with four branches and without absolute value. Explain how you arrived to your answer.

11. If $f(x)=\dfrac{x+1}{x}$ for $x\neq 0$ and $f(g(x))=x$ for $x\neq 1$, find $g\left(1-\dfrac{1}{x}\right)$ in terms of x. Explain how you arrived to your answer.

12. Comment on the truth of each of the following statements for the function $f(x) = x|x|$.

I. The function is even.
II. The graph of the function is symmetric with respect to the origin.
III. The function is neither even nor odd.

13. Given $f(x) = (x-9)^2 + 4$, $g(x) = \dfrac{x^2+14}{5-x^2}$, $h(x) = \dfrac{x-x^2}{x^2+5}$, and $k(x) = 11 + x^2$, which of these functions are even? Explain how you arrived to your answer.

14. In the table below, f and g have the property that $f(g(x)) = g(f(x)) = x$ for all x.

x	1	2	3	4
f	4	6	1	5
g	a	b	c	d

Find the value of a + d. Explain how you arrived to your answer.

15. Given $f(3) = 10$, $f(5) = 20$, and $f(x) = \dfrac{f(x-2)}{f(x-1)}$, find $f(0)$.

16. f(x), g(x) and h(x) are all continuous functions with domains of all real numbers. h(x) is an even function, f(x) is cubic, and neither f(x) nor g(x) is an even or odd function. Given the table below, find $h(1)$ if $(f(h(g)))(1) - 37 = -100$. Explain how you arrived to your answer.

x	0	1	3	4
f(x)	1	A	-26	-63
g(x)	-3	-1	B	5
h(x)	0	C	36	64

17. If $f(x) = \dfrac{2x+51}{5}$, explain how to find $f^{-1}(x)$ and the value of $f^{-1}(51)$.

18. $f(x) = Ax^3 + Bx^2 + Cx + D = x^3 - 10x^2 + Cx + D$ where C and D are integers. If the zeros of f are 2, r_1, and r_2, explain why the value of $r_1 + r_2 = 8$.

19. $f(x) = a|x+b| + c$ has minimum value at the point (1, -4) and an x-intercept 2. Find the value of f(3). Explain how you arrived to your answer.

20. Find the domain of $f(g(x))$ if $f(x) = \dfrac{\sqrt{x-1}}{x^2-4}$ and $g(x) = x^2 - 1$. Show your answer using interval notation. Explain how you arrived to your answer.

21. Find the equation of the vertical asymptote(s) of $f(x)=\frac{3x^3+6x^2+3x}{x^2-1}$. Explain how you arrived to your answer.

22. The graph $h(x)=\frac{3x^2-12x}{x^3-16x}$ will contain a hole at what point(s)? Explain how you arrived to your answer.

23. Comment on the truth of each statement below.

I. $f(x)=\frac{x}{x-1}$ is an odd function.

II. $g(x)=\frac{x^2}{x^2+1}$ is an even function.

III. $f(x)=x^3-x^2+1$ is an odd function.

Solution: I. f(x) is neither even nor odd since $f(-x)=\frac{-x}{-x-1}\neq f(x)$ and $\frac{-x}{-x-1}\neq -f(x)$.

II. g(x) is even since $g(-x)=g(x)$.

III. $h(x)$ is neither even nor odd since only the cubed term would change signs when substituting –x.

24. Identify the domain of $f(x)=\frac{\sqrt{x+7}}{x+5}$ using interval notation. Explain how you arrived to your answer.

SWSC Functions Oral Reference – Contest 4
Sample Oral Questions w/Solutions

1. Find the polynomial equation of the cubic function f(x) which has zeros 4 and $3+\sqrt{2}$. Explain how you arrived to your answer.

   Solution: The given zeros yield factors of $x-4,\ x-3-\sqrt{2},\ x-3+\sqrt{2}$. Putting these factors together yields $f(x)=(x-4)\left((x-3)-\sqrt{2}\right)\left((x-3)+\sqrt{2}\right)=(x-4)\left((x-3)^2-2\right)=(x-4)\left(x^2-6x+7\right)$.
   Continuing the multiplication yields $f(x)=x^3-10x^2+31x-28$.

2. Two of the roots of the function $g(x)=x^3+3x^2+kx-12$ are real and unequal but have the same absolute value. Find the value of k. Explain how you arrived to your answer.

   Solution: Let the roots be a, –a, and b. Since the sum of the roots is –3, b = -3. Since the product of the roots is 12, then $3a^2=12 \rightarrow a=\pm 2$. Substituting any of the roots (-2, -3, or 2) into the function when g(x) = 0 yields k = –4.

3. If $f(x)=x^3-10x^2+Ax+B$, f has one root at x = 2, and A + B = 9, find the value of f(1). Explain how you arrived to your answer.

   Solution: Since $f(2)=0,\ 8-40+2A+B=0$ or $2A+B=32$. Solving the system $\begin{cases} A+B=9 \\ 2A+B=32 \end{cases}$, we obtain A = 23 and B = –14. Therefore, $f(x)=x^3-10x^2+23x-14$ and $f(1)=1^3-10(1)^2+23(1)-14=0$

4. For what values of x is f(x) > 0 if $f(x)=x^3-7x+6$? Express your answer using interval notation. Explain how you arrived to your answer.

   Solution: Using synthetic division $f(x)$ can be rewritten as $f(x)=(x-1)(x+3)(x-2)$. The function is positive on $(-3,1)\cup(2,\infty)$.

5. Through what quadrant(s) does the slant asymptote of $g(x)=\dfrac{x^2+6x+1}{x+2}$ pass through if it is graphed in the Cartesian coordinate system? Explain how you arrived to your answer.

   Solution: Using either synthetic division or polynomial division, $\dfrac{x^2+6x+1}{x+2}$ can be rewritten as $x+4-\dfrac{7}{x+2}$. The slant asymptote is the line formed by the quotient, neglecting the remainder. Therefore, the slant asymptote is $y=x+4$. The graph of $y=x+4$ passes through quadrants I, II and III.

6. If the graph of $f(x)=\dfrac{x^3+6x^2+3x-10}{x^2-6x+5}$ contains 'a' vertical asymptotes, 'b' horizontal asymptotes, and 'c' slant asymptotes, find the values of a, b, and c. Explain how you arrived to your answer.

Solution: $f(x)=\dfrac{x^3+6x^2+3x-10}{x^2-6x+5}=\dfrac{(x-1)(x+5)(x+2)}{(x-1)(x-5)}$. There is one vertical asymptote at x = 5. Since the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote. Instead, there is a slant asymptote (y = x + 12). Therefore a = 1, b = 0, and c = 1.

7. $f(x+1)=\dfrac{x^2+9x+12}{2x+7}$. If $f(3-x)$ is written in the form $f(3-x)=\dfrac{Ax^2+Bx+C}{Ax+B}$, find $f(3-x)$. Explain how you arrived to your answer.

Solution: $3-x=(-x+2)+1\rightarrow f(-x+2)=\dfrac{(-x+2)^2+9(-x+2)+12}{2(-x+2)+7}=\dfrac{x^2-13x+34}{-2x+11}$

8. For what values of x is the function $h(x)<0$ if $h(x)=\dfrac{3x}{x(x^2-16)}$? Express your answer using interval notation, and explain how you arrived to your answer.

Solution: $h(x)<0$ when $(x-4)(x+4)<0$. On $(-\infty,-4)$, $h(x)>0$. On $(-4,0)$, $h(x)<0$. At x = 0, h(x) is undefined. On (0, 4), $h(x)<0$. On $(4,\infty)$, $h(x)>0$. Therefore, $h(x)<0$ on $(-4,0)\cup(0,4)$.

9. If $g(x)=\dfrac{x^3-3x^2-9x+27}{x^2+2x-15}$:

a. Find the coordinates of the hole in the graph.
b. Find the number of zeros of g.
c. Find the x-intercept of the slant asymptote

Explain how you arrived to each answer.

Solution: $g(x)=\dfrac{x^3-3x^2-9x+27}{x^2+2x-15}=\dfrac{(x-3)(x^2-9)}{(x+5)(x-3)}=\dfrac{(x-3)^2(x+3)}{(x+5)(x-3)}$

a. There will be a hole in the graph of g(x) at (3, 0).
b. There is one zero of g: (-3, 0).
c. $\dfrac{(x-3)^2(x+3)}{(x+5)(x-3)}=\dfrac{(x-3)(x+3)}{(x+5)}=\dfrac{x^2-9}{x+5}$. Either by synthetic division or long division, $\dfrac{x^2-9}{x+5}=x-5+\dfrac{16}{x+5}$. The slant asymptote is y = x – 5. The x-intercept of the slant asymptote is 0 = x – 5 or x = 5.

10. Rewrite the function $f(x)=\left||x-3|-4\right|$ as a piecewise function with four branches and without absolute value. Explain how you arrived to your answer.

Solution: The expression inside $|x-3|$ will change sign when x – 3 = 0 or x = 3. The expression inside $\left||x-3|-4\right|$ will change sign when $|x-3|-4=0 \rightarrow |x-3|=4 \rightarrow \begin{cases} x-3=4 \\ x-3=-4 \end{cases} \rightarrow$ x = 7 and x = –1. This yields four intervals for the piecewise function: $x \le -1,\ -1 \le x \le 3,\ 3 \le x \le 7,\ x \ge 7$. Rewriting f(x) as a piecewise function yields $f(x)=\begin{cases} -x-1 & x \le -1 \\ x+1 & -1 \le x \le 3 \\ 7-x & 3 \le x \le 7 \\ x-7 & x \ge 7 \end{cases}$.

11. If $f(x)=\frac{x+1}{x}$ for $x \ne 0$ and $f(g(x))=x$ for $x \ne 1$, find $g\left(1-\frac{1}{x}\right)$ in terms of x. Explain how you arrived to your answer.

Solution: $f(x)=1+\frac{1}{x}$, so $1-\frac{1}{x}=f\left(g\left(1-\frac{1}{x}\right)\right)=1+\frac{1}{g\left(1-\frac{1}{x}\right)} \rightarrow \frac{-1}{x}=\frac{1}{g\left(1-\frac{1}{x}\right)} \rightarrow g\left(1-\frac{1}{x}\right)=-x$

Alternative Solution: $x=\frac{g(x)+1}{g(x)} \rightarrow g(x)=\frac{1}{x-1} \rightarrow g\left(1-\frac{1}{x}\right)=\frac{1}{1-\frac{1}{x}-1}=-x$

12. Comment on the truth of each of the following statements for the function $f(x)=x|x|$.

I. The function is even.
II. The graph of the function is symmetric with respect to the origin.
III. The function is neither even nor odd.

Solution: $f(-x)=-x|-x|=-x|x|$. Therefore, the function is not even since $f(-x)\ne f(x)$. The function is symmetric with respect to the origin since $f(-x)=-f(x)$. This also shows that the function is odd.

13. Given $f(x)=(x-9)^2+4$, $g(x)=\frac{x^2+14}{5-x^2}$, $h(x)=\frac{x-x^2}{x^2+5}$, and $k(x)=11+x^2$, which of these functions are even? Explain how you arrived to your answer.

Solution: An even function has the property that $m(-x)=m(x)$. The function f can be rewritten as $f(x)=x^2-18x+85$. $f(-x)=x^2+18x+85\ne f(x)$. $g(-x)=\frac{x^2+14}{5-x^2}$, $h(-x)=\frac{-x-x^2}{x^2+5}$, and $k(-x)=11+x^2$. Therefore, g and k are even functions.

14. In the table below, f and g have the property that $f\left(g\left(x\right)\right)=g\left(f\left(x\right)\right)=x$ for all x.

x	1	2	3	4
f	4	6	1	5
g	a	b	c	d

Find the value of a + d. Explain how you arrived to your answer.

Solution: f and g are inverses, so (r, s) on f will match to (s, r) on g. Since f contains (3, 1), g will contain (1, 3). Therefore a = 3. Since f contains (1, 4), g will contain (4, 1). Therefore d = 1. a + d = 3 + 1 = 4.

15. Given $f\left(3\right)=10$, $f\left(5\right)=20$, and $f\left(x\right)=\frac{f\left(x-2\right)}{f\left(x-1\right)}$, find $f\left(0\right)$.

Solution: If x = 5, $f\left(5\right)=\frac{f\left(3\right)}{f\left(4\right)}=20$. Since $f\left(3\right)=10$, $\frac{10}{f\left(4\right)}=20\rightarrow f\left(4\right)=\frac{1}{2}$. $f\left(4\right)=\frac{f\left(2\right)}{f\left(3\right)}\rightarrow$ $\frac{1}{2}=\frac{f\left(2\right)}{10}\rightarrow f\left(2\right)=5$. $f\left(3\right)=\frac{f\left(1\right)}{f\left(2\right)}\rightarrow 10=\frac{f\left(1\right)}{5}\rightarrow f\left(1\right)=50$. $f\left(2\right)=\frac{f\left(0\right)}{f\left(1\right)}\rightarrow 5=\frac{f\left(0\right)}{50}\rightarrow f\left(0\right)=250$.

16. f(x), g(x) and h(x) are all continuous functions with domains of all real numbers. h(x) is an even function, f(x) is cubic, and neither f(x) nor g(x) is an even or odd function. Given the table below, find $h\left(1\right)$ if $\left(f\left(h\left(g\right)\right)\right)\left(1\right)-37=-100$. Explain how you arrived to your answer.

x	0	1	3	4
f(x)	1	A	-26	-63
g(x)	-3	-1	B	5
h(x)	0	C	36	64

Solution: $g\left(1\right)=-1\rightarrow f\left(h\left(-1\right)\right)-37=-100\rightarrow f\left(h\left(-1\right)\right)=-63$. Since h is even, $f\left(h\left(-1\right)\right)=f\left(h\left(1\right)\right)=-63\rightarrow f\left(C\right)=-63\rightarrow C=4$. Therefore, h(1) = 4.

17. If $f\left(x\right)=\frac{2x+51}{5}$, explain how to find $f^{-1}\left(x\right)$ and the value of $f^{-1}\left(51\right)$.

Solution: To find $f^{-1}\left(x\right)$, we must solve $x=\frac{2f^{-1}\left(x\right)+51}{5}$ for $f^{-1}\left(x\right)$. Solving, we obtain $f^{-1}\left(x\right)=\frac{5x-51}{2}$. The value of $f^{-1}\left(51\right)=\frac{5\left(51\right)-51}{2}=102$.

18. $f(x) = Ax^3 + Bx^2 + Cx + D = x^3 - 10x^2 + Cx + D$ where C and D are integers. If the zeros of f are 2, r_1, and r_2, explain why the value of $r_1 + r_2 = 8$.

Solution: The sum of the zeros of the function is $\frac{-B}{A} = \frac{-(-10)}{1} = 10$. Therefore, $r_1 + r_2 + 2 = 10$ or $r_1 + r_2 = 8$.

19. $f(x) = a|x + b| + c$ has minimum value at the point (1, -4) and an x-intercept 2. Find the value of f(3). Explain how you arrived to your answer.

Solution: Since the minimum occurs at x = 1, the cusp occurs at x = 1. Therefore b = –1. Since at x = 1, f(x) = –4, so c = –4. At y = 0, x = 2, so a = 4. Therefore $f(x) = 4|x - 1| - 4 \rightarrow f(3) = 4$.

20. Find the domain of $f(g(x))$ if $f(x) = \frac{\sqrt{x-1}}{x^2 - 4}$ and $g(x) = x^2 - 1$. Show your answer using interval notation. Explain how you arrived to your answer.

Solution: $f(g(x)) = \frac{\sqrt{(x^2-1)-1}}{(x^2-1)^2 - 4}$. Since the radicand must be non-negative, $x^2 - 1 - 1 \geq 0 \rightarrow x^2 \geq 2 \rightarrow |x| \geq \sqrt{2}$. In addition, $(x^2-1)^2 - 4 \neq 0 \rightarrow (x^2-1)^2 \neq 4 \rightarrow x^2 - 1 \neq \pm 2 \rightarrow$ $x^2 \neq 3, x^2 \neq -1 \not\rightarrow x \neq \pm\sqrt{3}$. Thus, the domain is $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}) \cup (\sqrt{3}, \infty)$.

21. Find the equation of the vertical asymptote(s) of $f(x) = \frac{3x^3 + 6x^2 + 3x}{x^2 - 1}$. Explain how you arrived to your answer.

Solution: $f(x) = \frac{3x^3 + 6x^2 + 3x}{x^2 - 1} = \frac{3x(x^2 + 2x + 1)}{(x-1)(x+1)} = \frac{3x(x+1)^2}{(x-1)(x+1)}$. Vertical asymptotes exist for x values which make the denominator equal to zero, unless those factors are eliminated with the numerator. Therefore, x = 1 is the only vertical asymptote.

22. The graph $h(x) = \frac{3x^2 - 12x}{x^3 - 16x}$ will contain a hole at what point(s)? Explain how you arrived to your answer.

Solution: Since $h(x) = \frac{3x(x-4)}{x(x-4)(x+4)}$, there will be a hole when x = 0 and x = 4. The expression simplifies to $\frac{3}{x+4}$, indicating holes at $\left(0, \frac{3}{4}\right)$ and $\left(4, \frac{3}{8}\right)$.

23. Comment on the truth of each statement below.

I. $f(x)=\frac{x}{x-1}$ is an odd function.

II. $g(x)=\frac{x^2}{x^2+1}$ is an even function.

III. $f(x)=x^3-x^2+1$ is an odd function.

Solution: I. f(x) is neither even nor odd since $f(-x)=\frac{-x}{-x-1}\neq f(x)$ and $\frac{-x}{-x-1}\neq -f(x)$.

II. g(x) is even since $g(-x)=g(x)$.

III. $h(x)$ is neither even nor odd since only the cubed term would change signs when substituting –x.

24. Identify the domain of $f(x)=\frac{\sqrt{x+7}}{x+5}$ using interval notation. Explain how you arrived to your answer.

Solution: Since the denominator cannot be zero, $x+5\neq 0\rightarrow x\neq -5$. In addition, the sum under the radical must be greater than or equal to zero, $x+7\geq 0\rightarrow x\geq -7$. In all, the domain is $[-7,-5)\cup(-5,\infty)$.